K22P 0190

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, April 2022
(2018 Admission Onwards)
MATHEMATICS
MAT 2C 07 : Measure and Integration

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. Show that if $m^*(E) = 0$, then E is measurable.
2. Show that there exists an uncountable set with measure zero.
3. Give an example of a function which is Lebesque integrable but not Riemann integrable.
4. Prove that if f and g are integrable functions, then f + g is also integrable.
5. Define $L^p(\mu)$ and prove that if f, g $\in L^p(\mu)$ and a, b are constants, then $af + bg \in L^p(\mu)$.
6. Define integral of a measurable simple function with respect to a measure μ.

(4×4=16)

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove that every interval is measurable.
   b) Prove that the class of all Lebesque measurable functions is a σ – algebra.
   c) Show that for any measurable function f and g.
   ess.sup. $(f + g) \leq$ ess.sup.f + ess.sup.g and give an example of strict inequality.

P.T.O.

8. a) Construct a non-measurable set.
   b) Let f be a measurable function and let f = g a.e, then prove that g is measurable.

9. a) State and prove Fatuous Lemma.
   b) Show that $\int_1^\infty \frac{dx}{x} = \infty$.

Unit – II

10. a) State and prove Lebesque Dominated convergence theorem.
    b) Let f be a bounded measurable function defined on the finite interval (a, b). Show that $\lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x \, dx = 0$.

11. a) Let μ^* be an outer measure of H(R) and let S^* denote the class of μ^* – measurable sets. Then prove that S^* is a σ – ring and μ^* restricted to S^* is a complete measure.
    b) Define a σ – finite measure. Show that if μ is a σ – finite measure on R, then the extension $\bar{\mu}$ of μ to S^* is also σ – finite.

12. a) Show that Lebesque measure is a σ – finite measure and complete.
    b) If μ is a σ – finite measure on a ring R, then prove that it has a unique extension to the σ – ring S(R).

Unit – III

13. a) Let $[\![X, S, \mu]\!]$ be a measure space and f a non-negative measurable function. Then prove that $\varphi(E) = \int_E f \, d\mu$ is a measure on the measurable space $[\![X, S]\!]$. Also prove that if $\int f \, d\mu < \infty$, then $\forall \in > 0, \exists \, \delta > 0$ such that, if $A \in S$ and $\mu(A) < \delta$, then $\varphi(A) < \delta$.
    b) Define $L^\infty(X, \mu)$ and prove that $L^\infty(X, \mu)$ is a vector space over the real numbers.

14. a) State and prove Hölder's inequality.
    b) State and prove Minkowski's inequality.

15. a) If $1 \le p \le \infty$ and $\{f_n\}$ is a sequence in $L^p(\mu)$ such that $\|f_n - f_m\|_p \to 0$ as m, n $\to \infty$, then prove that there exists a function f and a subsequence $\{n_i\}$ such that $\lim f_{n_i} = f$ a.e. Also prove that $f \in L^p(\mu)$ and $\lim \|f_n - f\|_p = 0$.
    b) Prove that L^∞ (μ) is a complete metric space. **(4×16=6**